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Invited Lecture

Some flow effects in continuum theory for smectic liquid crystals

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This paper presents a constrained theory for smectic C liquid crystals that may be useful for the analysis of some effects in these materials. The theory is based on two simplifying assumptions, namely that the layers although deformed remain of constant thickness, and also that the tilt with respect to the layer normal remains fixed. The equilibrium version of the theory proves to be a non-linear generalization of the earlier Orsay theory, and promises to model a number of static effects satisfactorily. Here our aim is to examine preliminary predictions based on the corresponding dynamic theory, where some progress proves possible for shear flow, and also for a shear wave reflection-refraction experiment useful for the measurement of some viscous coefficients.

1. Introduction

Recently Leslie *et al.* [1] have proposed a continuum theory for smectic C liquid crystals that may be useful for the analysis of certain effects in these materials. In order to avoid excessive mathematical complexity the theory appeals to two simplifying assumptions that clearly restrict its range of applicability. These are that the layers although deformable remain of constant thickness, and also that the angle of tilt of the molecular alignment with respect to the layer normal remains constant. The former certainly appears reasonable in many instances, and the latter also, provided pretransitional and thermal effects are not significant. Their theory is non-linear and is not restricted to small perturbations of planar layers.

The theory employs two directors to model smectic C configurations leading to a quadratic elastic energy that contains nine terms for non-chiral smectic C liquid crystals [2] and eleven terms for chiral materials [3]. In the particular case of small distortions of planar layers, this energy reduces identically to that proposed earlier by the Orsay Group [4] for such small disturbances. Initial studies of static configurations include successful analyses of Dupin [5] and parabolic cyclides [6] which go some way towards justifying the simplifications employed. More recent investigations of equilibrium phenomena include effects in cylindrical layers, either confined in a wedge-like gap [7] or between concentric cylinders [8]. The former shows that information concerning certain elastic constants may follow from such studies, while the latter examines walls induced by the application of a magnetic field, the number of walls that are possible depending upon the angle of inclination of the field to the cylinder axis. The indications are that once problems concerning uniform alignment of smectic C liquid crystals are overcome, progress with the static version of this theory may be rather good.

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Here, however, our aim is to examine preliminary predictions based on the corresponding dynamic theory. This theory contains twenty dissipative terms in the stress tensor, and is therefore proportionally more complex than the static theory. However, some progress is possible in discussing the simpler arrangements in shear flow, especially when the layers are planar and parallel to the bounding plates. In this particular case it is possible to consider both chiral and non-chiral materials, and to examine the influence of flow in terms of unwinding the helical twist in chiral smectic C liquid crystals. One can equally study the influence of flow upon the bookshelf geometry when the shear is within the layers, this analysis being confined to non-chiral smectics, but here the predictions appear to raise some questions. While we discuss these flow problems below, fuller details are available in a separate paper by Gill and Leslie [9] also presented at this Conference.

One can of course employ this non-linear theory to analyse situations that involve small distortions of planar layers, and below we consider one such problem that has relevance for viscosity measurements, namely the reflection and refraction of a shear wave at a solid–smectic interface. For one particular configuration the analysis is straightforward, but in others no solutions are available presumably because the actual response is incompatible with the constraints imposed.

2. Continuum theory

This section presents a brief account of the rather general constrained continuum theory proposed recently by Leslie *et al.* [1] to model simpler aspects of the behaviour of non-chiral smectic C liquid crystals. The extension to chiral smectic C materials is also indicated. This theory is constrained in that it excludes variations in the layer spacing thickness, and also changes in tilt with respect to the layer normal, but it may prove useful for the interpretation of certain observations and experiments for this class of liquid crystals.

The layers in a smectic liquid crystal are most readily described by a density wave vector **a**, but the assumption that the layer spacing remains constant implies that this vector can without loss of generality be identified with the unit normal to the layers. Following de Gennes [10] the theory employs a second unit vector **c** perpendicular to **a** to describe the direction of tilt with respect to the layer normal. Hence the two directors in the theory are subject to

$$\mathbf{a}\cdot\mathbf{a}=\mathbf{c}\cdot\mathbf{c}=1, \quad \mathbf{a}\cdot\mathbf{c}=0. \tag{2.1}$$

Also, as both Oseen [11] and de Gennes [10] argue, the vector **a** in the absence of defects in the layering is subject to the constraint

$$\operatorname{curl}\mathbf{a}=0. \tag{2.2}$$

A further constraint arises from the assumption of incompressibility, and thus the velocity vector **v** must satisfy

$$\operatorname{div}\mathbf{v}=0, \tag{2.3}$$

and as a consequence the density ρ is constant.

The balance laws of the theory are essentially those of classical continuum mechanics, namely the balances of linear and angular momentum. In cartesian tensor notation the former takes the familiar form

$$\rho\dot{v}_i=\rho F_i+t_{ij,j}, \tag{2.4}$$

F denoting the body force per unit mass, **t** the stress tensor, and the superposed dot the material time derivative. However, the latter includes terms commonly disregarded, and is

$$0=\rho K_i+e_{ijk}t_{kj}+l_{ij,j} \tag{2.5}$$

with **K** denoting the external body moment per unit mass and **l** the couple stress tensor, the inertial term being omitted on the grounds that it is in general negligible. Also thermal effects are ignored. In the above a repeated index is subject to the summation convention, a comma preceding an index denotes a partial derivative with respect to the corresponding spatial coordinate, and e_{ijk} is the alternator.

The constitutive relations for stress and couple stress are respectively

$$\left.\begin{aligned} t_{ij}&=-p\delta_{ij}+\beta_p e_{pjk}a_{k,i}-\frac{\partial W}{\partial a_{k,j}}a_{k,i}-\frac{\partial W}{\partial c_{k,j}}c_{k,i}+\tilde{t}_{ij},\\ l_{ij}&=\beta_p a_p\delta_{ij}-\beta_i a_j+e_{ipq}\left(a_p\frac{\partial W}{\partial a_{q,j}}+c_p\frac{\partial W}{\partial c_{q,j}}\right),\end{aligned}\right\} \tag{2.6}$$

where the pressure p and the vector $\boldsymbol{\beta}$ arise from the constraints in equations (2.3) and (2.2), respectively, while the energy W takes the form

$$\begin{aligned} 2W=&K_1^a(a_{i,i})^2+K_2^a(c_ia_{i,j}c_j)^2+2K_3^a a_{i,i}c_ja_{j,k}c_k\\ &+K_1^c(c_{i,i})^2+K_2^c c_{i,j}c_{i,j}+K_3^c c_{i,j}c_jc_{i,k}c_k\\ &+2K_4^c c_{i,j}c_jc_{i,k}a_k+2K_1^{ac}c_{i,i}c_ja_{j,k}c_k+2K_2^{ac}c_{i,i}a_{j,j},\end{aligned} \tag{2.7}$$

where the Ks are simply constants. The tensor $\tilde{\mathbf{t}}$ denotes the viscous stress which is the sum of a symmetric part

$$\begin{aligned} \tilde{t}^{s}_{ij}=&\mu_0D_{ij}+\mu_1a_pa_qD_{pq}a_ia_j+\mu_2(D_{ik}a_ka_j+D_{jk}a_ka_i)\\ &+\mu_3c_pc_qD_{pq}c_ic_j+\mu_4(D_{ik}c_kc_j+D_{jk}c_kc_i)+\mu_5c_pa_qD_{pq}(a_ic_j+a_jc_i)\\ &+\lambda_1(A_ia_j+A_ja_i)+\lambda_2(C_ic_j+C_jc_i)+\lambda_3c_pA_p(a_ic_j+a_jc_i)\\ &+\kappa_1(D_{ik}a_kc_j+D_{jk}a_kc_i+D_{ik}c_ka_j+D_{jk}c_ka_i)+\kappa_2[2a_pc_qD_{pq}a_ia_j\\ &+a_pa_qD_{pq}(a_ic_j+a_jc_i)]+\kappa_3[2a_pc_qD_{pq}c_ic_j+c_pc_qD_{pq}(a_ic_j+a_jc_i)]\\ &+\tau_1(C_ia_j+C_ja_i)+\tau_2(A_ic_j+A_jc_i)+2\tau_3c_pA_pa_ia_j+2\tau_4c_pA_pc_ic_j,\end{aligned} \tag{2.8}$$

and a skew-symmetric part

$$\begin{aligned} \tilde{t}^{ss}_{ij}=&\lambda_1(D_{jk}a_ka_i-D_{ik}a_ka_j)+\lambda_2(D_{jk}c_kc_i-D_{ik}c_kc_j)+\lambda_3c_pa_qD_{pq}(a_ic_j-a_jc_i)\\ &+\lambda_4(A_ja_i-A_ia_j)+\lambda_5(C_jc_i-C_ic_j)+\lambda_6c_pA_p(a_ic_j-a_jc_i)\\ &+\tau_1(D_{jk}a_kc_i-D_{ik}a_kc_j)+\tau_2(D_{jk}c_ka_i-D_{ik}c_ka_j)+\tau_3a_pa_qD_{pq}(a_ic_j-a_jc_i)\\ &+\tau_4c_pc_qD_{pq}(a_ic_j-a_jc_i)+\tau_5(A_jc_i-A_ic_j+C_ja_i-C_ia_j),\end{aligned} \tag{2.9}$$

where

$$\left.\begin{aligned} 2D_{ij}&=v_{i,j}+v_{j,i}, & 2W_{ij}&=v_{i,j}-v_{j,i},\\ A_i&=\dot{a}_i-W_{ij}a_j, & C_i&=\dot{c}_i-W_{ij}c_j,\end{aligned}\right\} \tag{2.10}$$

and the coefficients are constants.

The intrinsic viscous torque in equation (2.5) can conveniently be expressed as

$$e_{ijk}\tilde{t}^{ss}_{kj} = e_{ijk}(a_j\tilde{g}^a_k + c_j\tilde{g}^c_k), \tag{2.11}$$

where

$$\tilde{g}^a_i = -2(\lambda_1 D_{ik}a_k + \lambda_3 c_i c_p a_q D_{pq} + \lambda_4 A_i + \lambda_6 c_i c_p A_p$$
$$+ \tau_2 D_{ik}c_k + \tau_3 c_i a_p a_q D_{pq} + \tau_4 c_i c_p c_q D_{pq} + \tau_5 C_i), \tag{2.12}$$

$$\tilde{g}^c_i = -2(\lambda_2 D_{ik}c_k + \lambda_5 C_i + \tau_1 D_{ik}a_k + \tau_5 A_i), \tag{2.13}$$

and as a consequence the viscous dissipation inequality can be written as

$$\tilde{t}^s_{ij}D_{ij} - \tilde{g}^a_i A_i - \tilde{g}^c_i C_i \geqslant 0, \tag{2.14}$$

which imposes restrictions on the various viscous coefficients. Moreover, the relationship in equation (2.11) allows us to rewrite the balance of angular momentum in equation (2.5) as two equations

$$\left(\frac{\partial W}{\partial a_{i,j}}\right)_{,j} - \frac{\partial W}{\partial a_i} + \tilde{g}^a_i + \gamma a_i + \mu c_i + e_{ijk}\beta_{k,j} = 0, \tag{2.15}$$

and

$$\left(\frac{\partial W}{\partial c_{i,j}}\right)_{,j} - \left(\frac{\partial W}{\partial c_i}\right) + \tilde{g}^c_i + \tau c_i + \mu a_i = 0, \tag{2.16}$$

where γ, μ and τ are arbitrary multipliers. The body moment terms are omitted here since external moments are not considered in what follows. Also, the balance of linear momentum in equation (2.4) can be expressed more conveniently as

$$\rho\dot{v}_i = \rho F_i - \tilde{p}_{,i} + \tilde{g}^a_k a_{k,i} + \tilde{g}^c_k c_{k,i} + \tilde{t}_{ij,j} \tag{2.17}$$

where

$$\tilde{p} = p + W, \tag{2.18}$$

this using equations (2.15) and (2.16). The forms of equations (2.15), (2.16) and (2.17) are in many respects more convenient for the calculations that follow.

At this point it is perhaps helpful to make some remarks to clarify the physical role of the vector $\boldsymbol{\beta}$ associated with the constraint in equation (2.2). To this end we consider planar layers subject to some deformation of the c director, through flow or an external body moment, and choose cartesian axes with $\mathbf{a}$ and the z axis coincident. The moments acting on the plane of the layers are

$$l_i = l_{ij}a_j = \beta_p a_p a_i - \beta_i + e_{ipq}\left(a_p\frac{\partial W}{\partial a_{q,j}} + c_p\frac{\partial W}{\partial c_{q,j}}\right)a_j, \tag{2.19}$$

this using the latter of equations (2.6), or simply

$$l_x = -\beta_x + \ldots, l_y = -\beta_y + \ldots, l_z = 0 + \ldots, \tag{2.20}$$

where we only give the β contributions explicitly. From this we see that these terms provide torques acting upon the layers that can balance other moments and so maintain the assumed planar forms. More generally, if we assume a particular type of layering for geometrical or other reasons, the vector $\boldsymbol{\beta}$ can provide a mechanism through the couple stress to transmit torques needed to maintain the assumed configuration in equilibrium.

The extension of the above theory to chiral smectic C liquid crystals is relatively straightforward. For this class of liquid crystal we must add to the energy function in equation (2.7) two additional terms

$$W_c = K_4^a e_{ipq} c_p a_q a_{i,k} c_k + K_5^c e_{ipq} c_p a_q c_{i,k} a_k, \tag{2.21}$$

where the Ks are again constants. However, as Carlsson *et al.* [3] discuss, there is some doubt as to whether we should include the first of the above, since it gives rise to an equilibrium configuration not normally attributed to chiral smectic C liquid crystals. For this reason we omit it from our calculations. Lastly we note that the corresponding viscous stress for chiral smectic C is identical to that for non-chiral smectic C liquid crystals.

The above equations are invariant to the simultaneous change of sign in both directors **a** and **c**. However, for smectics with a material symmetry that is invariant to the independent change of sign of either **a** or **c**, the constitutive relations clearly simplify, the last three terms in the energy in equation (2.7), and the κ and τ terms in equations (2.8) and (2.9) being no longer acceptable.

3. Simple shear flow

In this section we consider two examples of shear flow that initially appear to be compatible with the layer structure assumed. In one case our conclusions are consistent with this assumption, but in the other the outcome is perhaps less clear.

First of all we consider a smectic confined between two parallel plates with the smectic layers everywhere parallel to the plates. The lower plate is at rest while the upper moves with a velocity V in a straight line in its own plane. Choosing cartesian axes with the z axis normal to the plates and the x axis parallel to the imposed velocity, it is natural to seek solutions of the equations of the previous section of the form

$$\mathbf{a} = (0, 0, 1), \quad \mathbf{c} = (\cos\phi(z), \sin\phi(z), 0), \quad \mathbf{v} = (u(z), v(z), 0). \tag{3.1}$$

Clearly this choice is consistent with the constraints of equations (2.1)–(2.3). Also, for this particular problem, our analysis covers both chiral and non-chiral materials.

For the above choice the equations of linear momentum reduce to

$$\left.\begin{aligned} \tilde{t}_{xz} &= (\eta_1 + \eta_2 \cos^2\phi) u' + \eta_2 \sin\phi \cos\phi v' = c_1, \\ \tilde{t}_{yz} &= (\eta_1 + \eta_2 \sin^2\phi) v' + \eta_2 \sin\phi \cos\phi u' = c_2, \end{aligned}\right\} \tag{3.2}$$

where the prime denotes differentiation with respect to z, c_1 and c_2 are constants, and the viscosities η_1 and η_2 are given by

$$2\eta_1 = \mu_0 + \mu_2 - 2\lambda_1 + \lambda_4, \quad 2\eta_2 = \mu_4 + \mu_5 + 2\lambda_2 - 2\lambda_3 + \lambda_5 + \lambda_6, \tag{3.3}$$

this assuming that the flow is due entirely to the relative motion of the plates, there being no imposed pressure gradients. Straightforwardly the equations (3.2) yield

$$\left.\begin{aligned} \eta_1(\eta_1 + \eta_2) u' &= c_1(\eta_1 + \eta_2 \sin^2\phi) - c_2 \eta_2 \sin\phi \cos\phi, \\ \eta_1(\eta_1 + \eta_2) v' &= c_2(\eta_1 + \eta_2 \cos^2\phi) - c_1 \eta_2 \sin\phi \cos\phi, \end{aligned}\right\} \tag{3.4}$$

giving the velocity gradients in terms of the alignment ϕ and the applied shears.

The equations for angular momentum reduce to

$$K_2^c \phi'' + (\tau_1 - \tau_5)(u' \sin\phi - v' \cos\phi) = 0, \tag{3.5}$$

this entailing a choice of the vector $\boldsymbol{\beta}$ of the form

$$\beta_x = \beta_1(z), \quad \beta_y = \beta_2(z), \quad \beta_z = 0. \tag{3.6}$$

However, employing equations (3.4), equation (3.5) becomes

$$K_2^c \phi'' + (\tau_1 - \tau_5)(c_1 \sin\phi - c_2 \cos\phi)/\eta_1 = 0, \tag{3.7}$$

which readily integrates to yield

$$K_2^c \phi'^2 + 2(\tau_5 - \tau_1)(c_1 \cos\phi + c_2 \sin\phi)/\eta_1 = c_3, \tag{3.8}$$

where c_3 is a constant.

The boundary conditions to be satisfied are

$$u(d) = V, \quad u(-d) = v(d) = v(-d) = 0, \quad \phi(d) = \phi_1, \quad \phi(-d) = \phi_2, \tag{3.9}$$

where $2d$ denotes the distance between the plates, the origin having been chosen midway between, and ϕ_1 and ϕ_2 are prescribed angles. Of particular interest with regard to chiral materials is the choice

$$\phi_1 = -\phi_2 = \phi_0 \tag{3.10}$$

with the twist ϕ an odd function of z. From equations (3.4) and the conditions (3.9), it quickly follows that

$$\left.\begin{aligned} \eta_1(\eta_1+\eta_2)V &= c_1 \int_{-d}^{d} (\eta_1 + \eta_2 \sin^2\phi)\, dz - c_2\eta_2 \int_{-d}^{d} \sin\phi\cos\phi\, dz, \\ 0 &= c_2 \int_{-d}^{d} (\eta_1 + \eta_2 \cos^2\phi)\, dz - c_1\eta_1 \int_{-d}^{d} \sin\phi\cos\phi\, dz. \end{aligned}\right\} \tag{3.11}$$

Naturally our choice of the constants c_1 and c_2 must be consistent with the above, and once the twist ϕ has been determined from equation (3.8), these equations essentially serve to determine c_1 and c_2 in terms of V, the gapwidth $2d$, and the material parameters. When the twist is odd, the latter of equations (3.11) clearly requires c_2 to be zero, and the former therefore reduces to

$$\eta_1(\eta_1+\eta_2)V = 2c_1 \int_0^d (\eta_1 + \eta_2 \sin^2\phi)\, dz, \tag{3.12}$$

which ultimately relates to c_1 to V.

The material parameters in the equations (3.8) and (3.11) appear in only three combinations, essentially a factor $(\tau_5 - \tau_1)/K_2^c\eta_1$ in the former and the two viscosities η_1 and η_2 in the latter. Consequently a numerical integration of these equations is relatively straightforward, with no need for an excessive selection of values for various coefficients. However, since Gill and Leslie [9] present fuller details in another paper, it suffices here to summarize their principal results. Primarily they discuss solutions in which the twist ϕ is either symmetric or asymmetric. For the former the behaviour is rather straightforward, the c director tending to flow align with the twist angle ϕ either tending to zero or π, dependent upon whether τ_5 is greater than or less than τ_1, in agreement with the earlier prediction by Leslie *et al.* [1]. For the asymmetric solutions the results are rather more interesting with the twist in the centre of the cell gradually unwinding with increasing shear rate, the flow causing the twist to move out to the bounding plates. This prediction appears to be consistent with the earlier observations by Pieranski *et al.* [12]. Gill and Leslie also discuss predictions for smectic C_M materials discussed recently by Brand and Pleiner [13], for which the analysis is a special case of the above.

Our second example considers the so-called bookshelf geometry with the imposed shear parallel to the smectic layers. Here, therefore, we consider solutions in which

$$\mathbf{a}=(0,1,0), \quad \mathbf{c}=(\cos\phi(z),0,\sin\phi(z)), \quad \mathbf{v}=(u(z),v(z),0), \tag{3.13}$$

which again clearly satisfy the constraints in equations (2.1)–(2.3). In this case, however, it is not possible to include a discussion of chiral materials in our analysis, since this would require the inclusion of a y dependence in the twist angle ϕ and also presumably in the flow components u and v, which is rather beyond the scope of this study.

For the above choice, in the absence of any imposed pressure gradients, the equations of linear momentum simply yield

$$\left.\begin{aligned}\tilde{t}_{xz}&=\mu_1(\phi)u'+\mu_2(\phi)v'\cos\phi=c_1,\\ \tilde{t}_{yz}&=\mu_3(\phi)v'+\mu_2(\phi)u'\cos\phi=c_2,\end{aligned}\right\} \tag{3.14}$$

where c_1 and c_2 are again constants, and the viscosity functions are given by

$$\left.\begin{aligned}2\mu_1(\phi)&=\mu_0+\mu_4+\lambda_5+2\lambda_2\cos2\phi+2\mu_3\sin^2\phi\cos^2\phi,\\ 2\mu_2(\phi)&=\kappa_1+\tau_1+\tau_2+\tau_5+2(\kappa_3+\tau_4)\sin^2\phi,\\ 2\mu_3(\phi)&=\mu_0+\mu_2+2\lambda_1+\lambda_4\\ &\quad+(\mu_4+\mu_5-2\lambda_2+2\lambda_3+\lambda_5+\lambda_6)\sin^2\phi.\end{aligned}\right\} \tag{3.15}$$

As above we can derive from equations (3.14) expressions for the velocity gradients in terms of the twist ϕ and the imposed shear stresses.

For this problem the equations of angular momentum reduce to

$$f(\phi)\phi''+\frac{1}{2}\frac{df}{d\phi}\phi'^2-(\lambda_5+\lambda_2\cos2\phi)u'-(\tau_1+\tau_5)v'\cos\phi=0, \tag{3.16}$$

where

$$f(\phi)=K_2^c+K_1^c\cos^2\phi+K_3^c\sin^2\phi, \tag{3.17}$$

but this appears to entail a choice of the vector $\boldsymbol{\beta}$ of the form

$$\beta_x=\beta_1(z)y, \quad \beta_y=0, \quad \beta_z=\beta_3(z)y, \tag{3.18}$$

and also the scalar γ such that

$$\gamma=\bar{\gamma}(z)-y\beta_1'(z), \tag{3.19}$$

the prime once more denoting differentiation with respect to z.

We do not pursue the above analysis at present for essentially two reasons. Firstly equations (3.14) and (3.16) involve rather more combinations of viscous and elastic coefficients than our previous example, which creates some problems in terms of assigning values to the various material constants. Secondly, and perhaps more significantly, the y dependence of the vector $\boldsymbol{\beta}$ does imply that the restraining torques on the smectic layers become rather large, suggesting that the system may relax by some mechanism not considered in the present analysis, the occurrence of a domain structure being one possibility. Clearly it would be of interest to have more experimental evidence available as to what actually happens in this case, including the influence of the induced transverse flow upon the layers.

4. Reflection and refraction of a shear wave at a solid–smectic interface

In this section we present a brief analysis for smectic C liquid crystals of a technique that allows the measurement of certain viscous coefficients, the one first used for nematics by Martinoty and Candau [14]. For this it suffices to consider a plane interface between an isotropic elastic solid and the liquid crystal, and also to assume that both the solid and liquid crystal are unbounded, since the other boundaries are of no consequence. For a smectic C liquid crystal we assume that initially it is uniformly aligned with the layers at an arbitrary angle to the interface, this necessarily restricting our analysis to non-chiral materials, and thus referred to appropriate cartesian axes consider

$$\mathbf{a}=\mathbf{a}^0=(0,\cos\theta,\sin\theta),\quad \mathbf{c}=\mathbf{c}^0=(0,\sin\theta,-\cos\theta),\tag{4.1}$$

where θ is a given acute angle, the z axis being normal to the interface and positive into the smectic, the origin in the interface.

The displacement in the solid satisfies the equations of linear, isotropic elasticity, and having due regard to the symmetry of the problem we consider a normally incident wave of the form

$$u_x=A\exp i(\omega t-kz),\quad u_y=u_z=0,\tag{4.2}$$

where A is a known constant amplitude. The constant frequency ω and the wave number k satisfy

$$\rho_s\omega^2=\mu_s k^2,\tag{4.3}$$

ρ_s and μ_s denoting the density and shear modulus of the elastic solid. The reflected wave is naturally taken to be

$$u_x=B\exp i(\omega t+kz),\quad u_y=u_z=0,\tag{4.4}$$

with B a constant to be determined.

It is reasonable to assume that the induced flow in the smectic takes a form similar to the motion in the solid, and moreover that this does not disturb the uniform alignment of the layers but does excite a corresponding small disturbance of the tilted alignment. Consequently in the equations describing the smectic we set

$$v_x=b\exp i(\omega t-kqz),\quad v_y=v_z=0,\tag{4.5}$$

and also

$$c_x=c\exp i(\omega t-kqz),\quad c_y=\sin\theta,\quad c_z=-\cos\theta,\tag{4.6}$$

with the constants b, c and q to be determined. Clearly these selections are consistent with the constraints of equations (2.1)–(2.3). At the interface we must of course have continuity of displacement or equivalently velocity, and also of forces and moments acting on the interface. Here, however, we do not appeal to continuity of moments directly, but rather employ strong anchoring [10], which can be regarded as a limiting case of a moment condition arising from an interfacial energy (see, for example, references in [15] and [16]). At the interface we therefore assume that the director $\mathbf{c}$ is prescribed and remains unchanged.

Consistent with the use of the equations of linear elasticity we assume that the disturbances to the smectic crystal are also small, and therefore neglect products of the quantities b and c in the relevant equations. In this event the equations of linear momentum reduce to

$$(2\rho i\omega+\eta k^2q^2)b-2k\omega\nu qc=0,\tag{4.7}$$

where

$$\eta = (\mu_0 + \mu_2 - 2\lambda_1 + \lambda_4)\sin^2\theta + (\mu_0 + \mu_4 - 2\lambda_2 + \lambda_5)\cos^2\theta$$
$$+ (\tau_1 + \tau_2 - \tau_5 - \kappa_1)\sin 2\theta, \tag{4.8}$$

and

$$v = (\tau_1 - \tau_5)\sin\theta + (\lambda_5 - \lambda_2)\cos\theta. \tag{4.9}$$

Also angular momentum simply yields

$$ikvqb - (2i\omega\lambda_5 + k^2q^2\sigma)c = 0, \tag{4.10}$$

with

$$\sigma = K_2^c \sin^2\theta - K_4^c \sin 2\theta + (K_2^c + K_3^c)\cos^2\theta. \tag{4.11}$$

Note that here it is necessary to choose the vector $\boldsymbol{\beta}$ of the form

$$\beta_x = 0, \quad \beta_y = \beta \exp i(\omega t - kqz), \quad \beta_z = 0, \tag{4.12}$$

β being a constant, but γ, μ and τ are all zero.

For non-trivial solutions for the unknowns b and c, equations (4.7) and (4.10) lead to a determinantal condition

$$(\eta q^2 + 2\rho i\xi)(\sigma q^2 + 2i\lambda_5\xi) - 2i\xi v^2 q^2 = 0, \tag{4.13}$$

where

$$\xi = \omega/k^2. \tag{4.14}$$

Alternatively this can be rewritten as a quadratic in q^2,

$$\eta\sigma q^4 + 2i\xi q^2(\chi + \rho\sigma) - 4\rho\lambda_5\xi^2 = 0, \tag{4.15}$$

where

$$\chi = \lambda_5\eta - v^2. \tag{4.16}$$

Here, it seems reasonable to assume that

$$\rho\sigma \ll \chi, \tag{4.17}$$

the elastic coefficients presumably small compared with the viscous coefficients. With this assumption the roots of equation (4.15) are approximately

$$q_1^2 = -2i\xi\chi/\eta\sigma, \quad q_2^2 = -2i\xi\rho\lambda_5/\chi, \tag{4.18}$$

which provide two values of q with negative imaginary part, necessary to ensure bounded solutions. Corresponding to the above values we find that b and c are related by

$$\chi b_1 = ik\sigma v q_1 c_1, \quad \rho v b_2 = ik\chi q_2 c_2, \tag{4.19}$$

respectively, these again invoking the approximation (4.17).

Given that two modes are possible, the interfacial conditions yield

$$\left.\begin{aligned} &i\omega(A+B) = b_1 + b_2, \quad c_1 + c_2 = 0, \\ &2\mu_s(A-B) = \eta(q_1b_1 + q_2b_2), \end{aligned}\right\} \tag{4.20}$$

from which it follows that

$$2\mu_s(A-B)/(A+B) = i\omega\eta(q_1b_1 + q_2b_2)/(b_1 + b_2), \tag{4.21}$$

this reducing with the aid of the approximation in (4.17) to

$$(A-B)/(A+B) = iq_2\chi/2\rho_s\lambda_5\xi \tag{4.22}$$

with help from equations (4.3), (4.14), (4.18), (4.19) and (4.20).

The above clearly allows a measurement of the quantity χ/λ_5 or $\eta - \nu^2/\lambda_5$, and, by varying the angle of inclination of the layers to the interface, we can measure different combinations of the viscous coefficients. As Gill and Leslie [17] show, it is possible to repeat the above calculation for oblique incidence, providing some further opportunity for such measurements.

In the above the displacement induced in the solid is parallel to the interface but perpendicular to the plane of symmetry of the smectic C. However, if the displacement is not perpendicular to this plane of symmetry, there are apparently in general no simple solutions of the present type for the equations described in section two. This presumably indicates that the resulting deformation of the smectic is incompatible with the constraints imposed by the present theory, but we do not pursue this question here.

5. Concluding remarks

In view of the above, it is of interest to examine briefly small disturbances to uniform smectic layering. To this end consider perturbations to the layer normal $\mathbf{a}^0$ of the form

$$\mathbf{a} = \mathbf{a}^0 + \varepsilon\boldsymbol{\alpha} \exp i(\omega t - \mathbf{k}\cdot\mathbf{x}), \tag{5.1}$$

where $\boldsymbol{\alpha}$ and $\mathbf{k}$ are arbitrary constant vectors, ω a constant, and ε a small parameter. It follows at once from the constraint in equation (2.1) that

$$\mathbf{a}^0\cdot\boldsymbol{\alpha} = 0, \tag{5.2}$$

and from the additional constraint in equation (2.2) that

$$\mathbf{k} \wedge \boldsymbol{\alpha} = 0. \tag{5.3}$$

Thus, if the vector $\boldsymbol{\alpha}$ is not zero, it must be parallel to the wave vector $\mathbf{k}$ as well as normal to $\mathbf{a}^0$. This of course requires that the wave vector be normal to the unperturbed layer normal. Thus the constraints do rather curtail the perturbations that are possible. Such considerations may well have relevance to the above phenomenon, and certainly to analyses of light scattering based on the theory presented in this paper.

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